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TWO CLASSES OF MOTION OF THE KOVALEVSKAIA TOP*

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The motions of the Kovalevskaia top /1/ is studied for the case when the ultraelliptical integrals degenerate to the elliptical integrals /2-6/. The Appel'rot terminology is used to investigate the fourth class of simplest motions of the body,and one particular case belonging to the third class. The mathematical apparatus developed by Kovalevskaia is not used directly. An elementary transformation of the initial Euler-Poisson equations is used as the basis of the investigation. This yields explicit relations connecting all unknown variables with time, and these in turn are used to show the general laws governing the motions. The motions can be divided into the oscillatory and asymptotic motions, depending on the initial parameters. The Bobylev-Steklov motion or its particular case of a body rotating about a fixed axis as a physical pendulum, represents the limiting mode for all asymptotic motions.

1. Transformation of equations. Under the Kovalevskaia conditions the Euler-Poisson equations and their algebraic first integrals are usually written in the form

$$2\frac{dp}{dt}=qr,\quad 2\frac{dq}{dt}=-rp-\gamma'',\quad \frac{dr}{dt}=\gamma' \tag{1.1}$$

$$\frac{d\gamma}{dt}=\gamma'r-\gamma''q,\quad \frac{d\gamma'}{dt}=\gamma''p-\gamma r,\quad \frac{d\gamma''}{dt}=\gamma q-\gamma'p$$

$$\begin{gathered}2(p^2+q^2)+r^2-2\gamma=6l_1,\ 2(p\gamma+q\gamma')+r\gamma''=2l,\ \gamma^2+\gamma'^2+\gamma''^2=\\ (p^2-q^2+\gamma)^2+(2pq+\gamma')^2=k^2\end{gathered} \tag{1.2}$$

To start, we carry out the following linear change of the phase variables:

$$\begin{gathered}x_1=p\sqrt{2}-a,\quad y_1=q\sqrt{2},\quad z_1=r/\sqrt{2},\quad \alpha_1=\gamma+ap\sqrt{2}-a^2\\ \beta_1=\gamma'+aq\sqrt{2},\quad \gamma_1=\gamma''+ar/\sqrt{2},\quad \tau=t/\sqrt{2}\end{gathered} \tag{1.3}$$

where a is an arbitrary parameter. In the new variables the system (1.1) and the integrals (1.2) can be written in the form

$$\frac{dx_1}{d\tau}=y_1z_1,\quad \frac{dy_1}{d\tau}=-z_1x_1-\gamma_1,\quad \frac{dz_1}{d\tau}=\beta_1-ay_1 \tag{1.4}$$

$$\frac{d\alpha_1}{d\tau}=2z_1\beta_1-\gamma_1y_1,\quad \frac{d\beta_1}{d\tau}=-2z_1\alpha_1+\gamma_1x_1-a^2z_1,\quad \frac{d\gamma_1}{d\tau}=\alpha_1y_1-\beta_1x_1$$

$$\begin{gathered}{}^1/_2(x_1^2+y_1^2)+z_1^2-\alpha_1+2ax_1=h_1,\ (\alpha_1+a^2)x_1+\beta_1y_1+\gamma_1z_1-\\ {}^1/_2a(x_1^2+y_1^2)=m_1;\ \alpha_1^2+\beta_1^2+\gamma_1^2+a^2\alpha_1-{}^1/_2a^2(x_1^2+y_1^2)=E_1\\ \alpha_1(x_1^2-y_1^2)+2\beta_1x_1y_1-\gamma_1^2+{}^1/_4(x_1^2+y_1^2)^2+a^2x_1^2=K_1\\ h_1=3l_1-{}^1/_2a^2,\quad m_1=l\sqrt{2}+3al_1-{}^1/_2a^3,\quad E_1=1+2al\sqrt{2}+3l_1a^2-\\ {}^1/_2a^4,\ K_1=k^2-1-2al\sqrt{2}-3l_1a^2+{}^1/_4a^4\end{gathered} \tag{1.5}$$

Next we carry out the nonlinear transformation

$$\begin{gathered}x_2=-x_1M_1^{-1},\quad y_2=-y_1M_1^{-1},\quad z_2=z_1+2\gamma_1x_1M_1^{-1},\quad \gamma_2=\gamma_1M_1^{-1},\\ M_1=x_1^2+y_1^2\neq 0\\ \alpha_2=-\alpha_1+2(x_1^2-y_1^2)\gamma_1^2M_1^{-2}-2a^2x_1^2M_1^{-1},\\ \beta_2=-\beta_1+4x_1y_1\gamma_1^2M_1^{-2}-2a^2x_1y_1M_1^{-1}\end{gathered}$$

The transformation is symmetric and equations (1.4), (1.5) become

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$$\frac{dx_2}{d\tau} = y_2z_2, \quad \frac{dy_2}{d\tau} = -z_2x_2 - \gamma_2, \quad \frac{dz_2}{d\tau} = \beta_2 - 2m_1y_2, \quad \frac{d\gamma_2}{d\tau} = \alpha_2y_2 - \beta_2x_2 \tag{1.6}$$

$$\frac{d\alpha_2}{d\tau} = 2z_2\beta_2 - 4K_1\gamma_2y_2, \quad \frac{d\beta_2}{d\tau} = -2z_2\alpha_2 + 4K_1\gamma_2x_2 - a^2z_2$$

$$z_2^2 - \alpha_2 + 4m_1x_2 + 2K_1M_2 = h_1, \quad \alpha_2x_2 + \beta_2y_2 + \gamma_2z_2 + a^2x_2 = {}^1/_2a + m_1M_2 \tag{1.7}$$

$$\alpha_2(x_2^2 - y_2^2) + 2\beta_2x_2y_2 - \gamma_2^2 + a^2x_2^2 + K_1M_1^2 = {}^1/_4$$

$$\alpha_2^2 + \beta_2^2 + 4K_1\gamma_2^2 + a^2\alpha_2 - 2a^2K_1M_2 = E_1, \quad M_2 = x_2^2 + y_2^2$$

Let $m_1^2 + K_1^2 \neq 0$. We introduce another parameter λ, putting $\lambda^2 = 4K_1, a\lambda = 2m_1$. It is clear that real a and λ satisfying the above relations exist. We find that using the transformation

$$x_3\sqrt{2} = \lambda x_2 + a, \ y_3\sqrt{2} = \lambda y_2, \ z_3/\sqrt{2} = z_2, \ \beta_3 = \beta_2 - a\lambda y_2$$

$$\alpha_3 = \alpha_2 - a\lambda x_2, \ \gamma_3 = \lambda\gamma_2 - az_2, \ t_3 = \tau\sqrt{2}$$

we convert (1.6) into (1.1). Thus we see that the structure of the above three transformations does not alter the form of the Euler–Poisson equations. Assume that

$$m_1 = 0, \ K_1 = 0 \tag{1.8}$$

Since $dK_1/da = -2m_1$, the conditions adopted mean that the polynomial $K_1(a)$ has a multiple root. Eliminating a, we obtain a unique constraint imposed on the initial parameters. Using the conditions (1.8) we obtain, from (1.6) (neglecting for simplicity the indices accompanying the variables and denoting the differentiation with respect to τ by a dot)

$$\dot{x} = yz, \quad \dot{y} = -zx - \gamma, \ \dot{z} = \beta \tag{1.9}$$

$$\dot{\alpha} = 2z\beta, \ \dot{\beta} = -2z\alpha - a^2z, \quad \dot{\gamma} = \alpha y - \beta x$$

$$z^2 - \alpha = h_1, \ (\alpha + a^2)x + \beta y + \gamma z = {}^1/_2 z, \ \alpha^2 + \beta^2 + a^2\alpha = E_1$$

$$\alpha(x^2 - y^2) + 2\beta xy - \gamma^2 + a^2x^2 = {}^1/_4$$

2. Integration of the auxilliary system.

The phase variables $p, \ldots, \gamma''$ of the initial equations are known rational functions of the quantities $x_2, \ldots, \gamma_2$. Therefore under the conditions (1.8) the problem is reduced to that of finding the solutions of the system (1.9). We note that the set of equations

$$\dot{z} = \beta, \ \dot{\alpha} = 2z\beta, \ \dot{\beta} = -2z\alpha - a^2z \tag{2.1}$$

forms a closed subsystem. Both its integrals are already known: $z^2 - \alpha = h_1, \ \alpha^2 + \beta^2 + a^2z^2 = 1$. Let us clarify the dependence of z, α, β on time. Clearly,

$$\dot{z}^2 = -z^4 + (2h_1 - a^2)z^2 + 1 - h_1^2 = (z^2 - R_1)(R_2 - z^2)$$

$$R_{1,2} = h_1 - {}^1/_2a^2 \mp k, \ R_1R_2 = h_1^2 - 1$$

$$h_1 \leqslant 1, \ 0 \leqslant z^2 \leqslant R_1; \ h_1 > 1, \ 0 < R_1 \leqslant z^2 \leqslant R_2$$

For z to be real, it is necessary and sufficient that $R_2 > 0$. In what follows, we shall use a different form of the solution of the subsystem (2.1). Its integrals imply (φ is an auxilliary quantity)

$$\alpha + {}^1/_2a^2 = k\cos 2\varphi, \ \beta = -k\sin 2\varphi, \ \dot{\varphi} = z, \ \ddot{\varphi} = -k\sin 2\varphi$$

$$\dot{\varphi}^2 = h_1 - {}^1/_2a^2 + k\cos 2\varphi$$

Let us find x, y and γ, noting the important relation

$$\ddot{y} + (z^2 + \alpha)y = 0$$

We find that the above linear equation is satisfied by the expression

$$\Gamma = H_2\gamma - {}^1/_2\,az \quad (H_2 = h_1 - a^2)$$

and we have

$$\Gamma^2 + H_2H_3y^2 = {}^1/_4h_1N, \quad \dot{\Gamma}y - \Gamma\dot{y} = -{}^1/_4h_1 \tag{2.2}$$

$$H_3 = a^2h_1 - 1, \quad N = z^2 - H_2$$

Let us assume that $N > 0$. Then (2.2) yields

$$N^2\dot{\eta}^2 + H_2H_3\eta^2 = {}^1/_4h_1, \quad \eta = yN^{-1/2} \tag{2.3}$$

Having obtained η, we find

$$y = \eta N^{1/2}, \quad \Gamma = N^{3/2}\dot{\eta}, \quad H_2x + {}^1/_2a = -(H_2\beta y + \Gamma z)N^{-1} \tag{2.4}$$

Below we give another relationship for finding x, y and γ:

$$H_2^2 N_1^2 z^{-4} \xi'^2 + H_2 H_3 \xi^2 = h_1, \quad \xi = (x + {}^1/_2 a H_2^{-1}) N_1^{-1/2}$$
$$N_1 = z^2 - H_1 H_2^{-1} > 0, \quad H_1 = h_1^2 - 1 \tag{2.5}$$

We can now give a complete classification of the solutions depending onthe values of

N, N_1: 1°. $H_2 < 0, H_3 > 0$; 2°. $H_2 < 0, H_3 < 0$; 3°. $H_2 > 0, H_3 > 0$; 4°. $H_2 > 0, H_3 < 0, H_1 < 0$; 5°. $H_2 > 0, H_3 < 0, H_1 > 0$

(a special case where N and N_1 have opposite signs); 6°. $H_2 = 0$; 7°. $H_3 = 0$; 8°. $H_1 = 0$.

We have eight distinct cases. Equations (2.3) and (2.5) yield the solutions for the first four cases

1°. $$y = \frac{\varepsilon\mu}{2b} (z^2 - H_2)^{1/2} \operatorname{sh}\theta, \quad \Gamma = \frac{\varepsilon\mu}{2} (z^2 - H_2)^{1/2} \operatorname{ch}\theta, \quad \theta' = \frac{b}{z^2 - H_2} > 0$$
$$H_2 x + \frac{a}{2} = -\frac{\varepsilon\mu}{2b} (z^2 - H_2)^{-1/2} (H_2 \beta \operatorname{sh}\theta + b \operatorname{ch}\theta), \quad \mu = h_1^{1/2},$$
$$b = |H_2 H_3|^{1/2}, \quad \varepsilon = \pm 1$$

2°. $$H_1 < 0, \quad y = \frac{\mu}{2b} (z^2 - H_2)^{1/2} \sin\theta, \quad \Gamma = \frac{\mu}{2} (z^2 - H_2)^{1/2} \cos\theta$$
$$H_2 x + \frac{a}{2} = -\frac{\mu}{2} (z^2 - H_2)^{-1/2} \left(\frac{H_2}{b} \beta \sin\theta + z \cos\theta \right), \quad \theta' = \frac{b}{z^2 - H_2} > 0$$

3°. The calculations follow those of case 2°, but the sign of H_1 is different.

4°. $$x + \frac{a}{2H_2} = \frac{\varepsilon\mu}{b} \left(z^2 - \frac{H_1}{H_2} \right)^{1/2} \operatorname{sh}\theta, \quad \theta' = \frac{bz^2}{H_2 z^2 - H_1} > 0$$
$$y = \frac{\varepsilon\mu}{b} \left(z^2 - \frac{H_1}{H_2} \right)^{-1/2} \left(bz \frac{\operatorname{ch}\theta}{H_2} + \beta \operatorname{sh}\theta \right)$$
$$\Gamma = \frac{\varepsilon\mu}{b} \left(z^2 - \frac{H_1}{H_2} \right)^{-1/2} \left(-b\beta \frac{\operatorname{ch}\theta}{H_2} + H_3 z \frac{\operatorname{sh}\theta}{H_2} \right)$$

5°. Under these conditions both N and N_1 change their sign with time, therefore the equations (2.3) and (2.5) are no longer suitable. We shall use the relations (2.2), noting that

$$z^2 - H_2 = a^2 + \alpha, \quad \alpha = a_0^2 \cos^2\varphi - b_0^2 \sin^2\varphi, \quad a_0^2 = k + {}^1/_2 a^2, \quad b_0^2 = k - {}^1/_2 a^2$$

Let us write the first equation of (2.2) in the parametric form

$$\Gamma = \frac{\mu}{2} (a_0 \cos\varphi \operatorname{ch}\theta + b_0 \sin\varphi \operatorname{sh}\theta), \quad y = \frac{\mu}{2b} (a_0 \cos\varphi \operatorname{sh}\theta + b_0 \sin\varphi \operatorname{ch}\theta)$$
$$\frac{2b}{\mu} \left(H_2 x + \frac{a}{2} \right) = -H_2 b_0 \cos\varphi \operatorname{ch}\theta + H_1 a_0 \sin\varphi \operatorname{sh}\theta - z\theta' (a_0 \cos\varphi \operatorname{ch}\theta + b_0 \sin\varphi \operatorname{sh}\theta)$$

The second equation of (2.2) yields $\theta' = -a_0 b_0 (z + H_2^{1/2})^{-1}$. When $H_1 > 0$, the quantity z has a definite sign. We can assign the plus sign to it without loss of generality. We then find that the quantity θ' is negative and bounded.

6°. In this case the relation $z'' + (z^2 + \alpha) z = 0$, holds, and yields rapidly the values of x, y and γ

$$Lx = H_1 h_1^{-1} - \alpha - u^2, \quad Ly = -2\beta - 2zu, \quad L\gamma = zu^2 + 2\beta u - z^3 + zh_1^{-1}$$
$$La = 4H_1, \quad u = z^2 \geqslant 0$$

7°. In this case we have the relation $(\cos\varphi)'' + (z^2 + \alpha)\cos\varphi = 0$, and we obtain

$$Lx = \varepsilon a - h_1 z \cos\varphi - u \sin\varphi, \quad Ly = h_1 - u\cos\varphi, \quad L\gamma = -\varepsilon a z + \cos\varphi \quad L = 2\varepsilon (a^2 - h_1), \quad \varepsilon = \pm 1, \quad u' = \cos^2\varphi \geqslant 0$$

8°. The solution is obtained using the elementary methods.

$$Lx = a + zu, \quad Ly = \beta u z^{-1} + u'; \quad L\gamma = -u - \beta u' z^{-1} - az$$
$$u'^2 = u^2 - 1, \quad z'^2 = z^2 (a_1^2 - z^2), \quad a_1^2 = 2 - a^2, \quad L = 2 (a^2 - h_1)$$

3. Investigation of the motion.

The solution of (1.9) has been constructed. By virtue of the symmetrical character of the second transformation and the linearity of the first transformation, we can express the unknown variables $p, \dots, \gamma''$ very simply in terms of $r_2 \dots \gamma_2$. In the case of a quantitative investigation, it is sufficient to note that the expressions are rational functions with unique denominators which never vanish. The motions of the top can be separated into two types of motion. The first type motions are oscillatory and obey the condition $H_2H_3>0$. In all the remaining cases $H_2H_3\leqslant 0$ and the motions are asymptotic. The Bobylev — Steklov motion determines the limiting mode. Indeed, for the given conditions we find that $x_2\to\infty$, $y_2\to\infty$, $\gamma_2\to\infty$, and hence $p\to\text{const}$, $q\to 0$, $\gamma''+rp\to 0$ as $t\to\infty$, which characterizes only the motion in question. We shall indicate a connection with the classical study of Kovalevskaia. The conditions $m_1=0$, $K_1=0$ are the conditions for the roots of the polynomial $\varphi(s)$ to be multiple. The equations $H_\nu=0$ ($\nu=1,2,3$) construe the conditions of additional multiplicity of the roots of the polynomial $F(s)$.

4. Second class of motions. We consider the motion of a top under the constraints $l=0$, $3l_1=k$. This represents according to Appel'rot a particular variant of the simplest third class motions. Starting directly from the Euler—Poisson equations, we introduce three new variables ρ, φ and θ, as follows /7/:

$$2p=\rho\sin\theta,\quad r=\rho\cos\theta,\quad \gamma+p^2-q^2=k\cos 2\varphi,\quad \gamma'+2pq=-k\sin 2\varphi \tag{4.1}$$

Calculations yield

$$\theta^{\cdot}=q+k\rho^{-1}\sin\theta\sin 2\varphi,\quad 2\varphi^{\cdot}=\rho\cos\theta$$

Using the known integrals, we obtain the required equations for determining φ and θ

$$2\theta^{\cdot\cdot}\cos\theta+\theta^{\cdot 2}-3l_1\sin\theta\cos^2\theta+\frac{(9l_1^2-k^2)(2-1.5\sin^2\theta)}{3l_1+k\cos 2\varphi}+\frac{2l}{\rho}=0 \tag{4.2}$$

$$2\varphi^{\cdot 2}=\cos^2\theta\,(3l_1-k+2k\cos^2\varphi) \tag{4.3}$$

When $l=0$ and $3l_1=k$, equation (4.2) can be integrated to yield

$$\theta^{\cdot 2}=\cos\theta\,(\varepsilon-3l_1\cos\theta),\quad \varepsilon=\text{const} \tag{4.4}$$

Taking the integrals into account, we find that $\varepsilon^2=1$.

Let us indicate certain simple, but important inequalities. In the nontrivial case we have $l_1>0$. From (4.4) it follows that if $\varepsilon=+1$, then $\cos\theta\geqslant 0$, and if $\varepsilon=-1$, then $\cos\theta\leqslant 0$. Thus $\cos\theta$ retains its sign during the motion and this implies that r has a definite sign. We can put $i\geqslant 0$, i.e. $\varepsilon=+1$, without loss of generality. Let us now find φ. Assume that at some time $\rho=0$. Then $r=0$, $p=0$, $\gamma'=0$. Such initial conditions characterize a rotation about a fixed axis only, therefore we shall assume that $\rho\neq 0$. Then from $\rho^2=4k\cos^2\varphi$ it follows that $\cos\varphi\neq 0$. We can assume without loss of generality that $\cos\varphi>0$. Further, using the fact that $r=2\varphi^{\cdot}$, we find that $\varphi^{\cdot}\geqslant 0$. It follows that we can simplify (4.3) to obtain

$$\varphi^{\cdot}=\mu\cos\varphi\cos\theta,\quad \mu^2=k,\quad \mu>0$$

The substitution

$$\cos\varphi=\frac{1}{\operatorname{ch}\tau_1},\quad \sin\varphi=\operatorname{th}\tau_1,\quad \tau_1=\mu\int\cos\theta\,d\tau+\text{const}$$

is expedient.

5. Solution of the Euler—Poisson system. Let us inspect the properties of the angle θ. We write the equation (4.4) in the form $(2\mu\theta^{\cdot})^2+(1-2\mu^2\cos\theta)^2=1$. In the parametric form this equation becomes

$$2\mu\theta^{\cdot}=\sin\sigma,\quad 1-2\mu^2\cos\theta=\cos\sigma$$

We find that

$$\sigma^{\cdot}=-\mu\sin\theta,\quad 2\sigma^{\cdot\cdot}=-\sin\sigma\cos\theta,\quad 2\theta^{\cdot\cdot}=-\sin\theta\cos\sigma$$

and finally we obtain

$$2\theta_1^{\cdot\cdot}=-\sin\theta_1,\quad \theta_1^{\cdot 2}=\cos\theta_1+\mu^2\;(\theta_1=\theta+\sigma) \tag{5.1}$$

If we make the body rotate about a fixed horizontal axis coinciding with the axis of inertia y, making use of all energy possessed by the body in real motion, then the equation of motion of such a rotation is given by the last relation. We shall also show how to find τ_1:

$$\tau_1=-\frac{1}{2}\ln\Delta+\mu\int\frac{\mu^2+\theta_1^{\cdot 2}}{\Delta}\,dt,\quad \Delta=1+4\mu^2\theta_1^{\cdot 2}$$

Finally, we write the solution of the initial Euler—Poisson equations in the form

$$p=-\sigma^{\cdot}\cos\varphi,\quad q=\sigma^{\cdot}\sin\varphi+\theta^{\cdot},\quad r=2\varphi^{\cdot},\quad \sigma^{\cdot}=-\mu\sin\theta,\quad \theta^{\cdot}=\tfrac{1}{2}\mu^{-1}\sin\sigma$$

$$\varphi' = \mu \cos\theta \cos\varphi, \quad 2\mu^2 \cos\theta = 1 - \cos\sigma$$
$$\gamma = \cos\theta (\cos^2\varphi + \cos\sigma \sin^2\varphi) - \sin\theta \sin\sigma \sin\varphi$$
$$\gamma' = -\cos\varphi [\sin\theta \sin\sigma + (1 - \cos\sigma) \cos\theta \sin\varphi]$$
$$\gamma'' = \cos\theta \sin\sigma \sin\varphi + \sin\theta \cos\sigma$$

6. Properties of the motion.

The solution obtained makes it easy to establish the laws of motion of the real motion of the body. Making $t \to \infty$ we find that $\tau_1 \to \infty, \sin\varphi \to 1$, and from this follows $p \to 0, r \to 0, \gamma' \to 0$. The precession angle ψ is found from the expression $\psi' (\gamma^2 + \gamma'^2) = p\gamma + q\gamma'$. As $t \to \infty$, the precession rate $\psi' \to 0$, while the angle ψ itself tends to a finite limit. Thus, as $t \to \infty$ the motion of the body tends to a limiting motion, namely to the rotation about a fixed horizontal axis coinciding with the principal axis of inertia y, according to the law (5.1) of the physical pendulum.

We shall note another specific feature helpful in describing the pattern of motion. We know that $\cos\theta$ vanishes periodically. When it does vanish, we have $\theta' = 0$, $\cos\sigma = 1$ and hence $r = 0$, $\gamma = \gamma' = 0$. Furthermore the energy integral yields $p^2 + q^2 = 3l_1$. This means that at these particular instances the dynamic symmetry axis z becomes vertical and the angular velocity vector, which has at all these instances the same absolute value, becomes horizontal. This situation is repeated after every period of the pendulum (5.1).

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